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Large-order behaviour of the $1/N$ expansion in zero and one dimensions

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Abstract. The large-order behaviour of the $1/N$ expansion in the zero- and one-dimensional $g\phi^4$ model is investigated. The one-dimensional N-vector model is also considered. The asymptotic behaviour shows oscillations due to complex instantons. The phase and the amplitude of these leading behaviours are determined theoretically and are compared with the explicit numerical calculations of higher orders. In the one-dimensional $g\phi^4$ model the relevance of the large-N instanton is discussed.

1. Introduction

Recent studies of the perturbation expansion in the $g\phi^4$ model have led to the understanding that the instanton (or pseudo-particle) solution of the classical equations of motion characterises the large-order behaviour (Lam 1968, Lipatov 1977, Brézin *et al* 1977, Brézin and Parisi 1978). The method may be applied to various problems in field theories, for the Landau–Ginzburg–Wilson model, or for quantum mechanical problems like the one-dimensional anharmonic oscillator (Bender and Wu 1969).

In critical phenomena calculations, in addition to similar coupling constant or ϵ-expansions one also uses the $1/N$ expansion, in which N is the number of components of the order parameter. Several terms of this expansion for the critical exponents have been determined (Abe 1973); however, up to now, the nature of this expansion remains unknown.

In this paper we consider the $1/N$ expansion in the zero- and one-dimensional $g\phi^4$ model and also in the one-dimensional classical N-vector model. The general nature of this expansion in low dimensions may hold even in higher dimensions. Thanks to the smallness of space dimensionality, we have many methods to calculate the large-order behaviour. In the zero-dimensional $g\phi^4$ model we derive the expression for the large orders from an integral representation and from a differential equation. Previously, zero-and one-dimensional $g\phi^4$ models were considered in the $1/N$ expansion (Bray 1974a,b, Ferrell and Scalapino 1974), and a few terms were obtained. However, the large-order behaviour has not so far been investigated. We explicitly calculate the first 60 terms, and check that they agree with the theoretical asymptotic estimate.

In the one-dimensional N-vector model, the exact expression for the partition function is known. The large-order behaviour of the $1/N$ expansion for this solution may be related to the large-index behaviour of modified Bessel functions.

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In the one-dimensional $g\phi^4$ model we consider the equivalent quantum mechanical ground state energy problem. The $1/N$ expansion may then be related to a coupling constant expansion for a potential problem, but the potential is singular. However, there is an instanton solution for small negative coupling. Making analytic continuation of this large-order term, we obtain the large order behaviour for the positive-coupling case. Numerically we obtain the first 50 terms in the $1/N$ expansion.

2. Zero-dimensional $g\phi^4$ model

2.1 $1/N$ expansion

Counting the number of diagrams of a given expansion may be viewed as a field theory in which each propagator is set equal to unity. It thus corresponds to a theory in which space–time is reduced to one or to a finite number of points, i.e. a 'zero-dimensional' field theory. The counting of the diagrams of the perturbative expansion of the N-component $g\phi^4$ theory leads to the N-dimensional integral

$$I = \int \frac{\mathrm{d}^N x}{(2\pi)^{N/2}} \exp\left[-\left(\frac{1}{2}\boldsymbol{x}^2 + \frac{g}{N}(\boldsymbol{x}^2)^2\right)\right], \tag{2.1}$$

and after integration over the angles to

$$I = \frac{2^{1-N/2}}{\Gamma(N/2)} \int_0^\infty \mathrm{d}x\, x^{N-1} \exp\left[-\left(\frac{1}{2}x^2 + \frac{g}{N}x^4\right)\right]. \tag{2.2}$$

This function $I(g, N)$ is the generating function of the vacuum diagrams' counting. Expanded in powers of $1/N$ it leads to the number of modified vacuum diagrams which appear in the rules of the $1/N$ expansion (Ma 1976). By the change of variable $x = N^{1/2}\,\mathrm{e}^{t/2}$ we get the expression

$$I = \frac{2(N/2)^{N/2}}{\Gamma(N/2)} \int_{-\infty}^{\infty} \mathrm{e}^{-NF(t)}\,\mathrm{d}t, \tag{2.3}$$

where

$$F(t) = \tfrac{1}{2}\mathrm{e}^t + g\,\mathrm{e}^{2t} - \tfrac{1}{2}t. \tag{2.4}$$

The $1/N$ expansion is obtained by expanding the exponent of equation (2.3) around its saddle point. The saddle point t_c is obtained as one of the solutions of

$$F'(t_c) = \tfrac{1}{2}\mathrm{e}^{t_c} + 2g\,\mathrm{e}^{2t_c} - \tfrac{1}{2} = 0. \tag{2.5}$$

The leading saddle point is the only real solution of this equation:

$$t_A = \log \frac{-1 + (1+16g)^{1/2}}{8g}. \tag{2.6}$$

There are other solutions, and for later convenience we note

$$t_B^{\pm} = \log \frac{1 + (1+16g)^{1/2}}{8g} \pm \mathrm{i}\pi. \tag{2.7}$$

Expanding $F(t)$ in equation (2.4) around t_A we obtain the $1/N$ expansion series

$$I=\frac{2(N/2)^{N/2}}{\Gamma(N/2)}\,e^{-NF(t_A)}\left(\frac{2\pi}{NF''(t_A)}\right)^{1/2}\left(1+\sum_1^\infty\frac{A_K}{N^K}\right), \tag{2.8}$$

with

$$F(t_A)=\frac{-1+(1+16g)^{1/2}}{32g}+\frac{1}{4}-\frac{1}{2}\ln\frac{(1+16g)^{1/2}-1}{8g}. \tag{2.9}$$

2.2. Calculation of the large-order term

From the explicit integral representation (2.3) it is easy to calculate the large-order behaviour of the A_K's in equation (2.8). Performing the change of variable

$$F(t)-F(t_A)=f^2 \tag{2.10}$$

we are led to an integral of the type

$$J=\int_{-\infty}^{\infty}e^{-Nf^2}\left(\frac{dt}{df}\right)df. \tag{2.11}$$

The $1/N$ expansion of J is generated by expanding dt/df around $f=0$. The large-order behaviour of this expansion is related to the saddle points $t_B^\pm$, which are singularities of dt/df. We expand the quantity f around these $t_B^\pm$ as

$$f=f(t_B)+\tfrac{1}{2}f''(t_B)(t-t_B)^2+\dots$$

and we get

$$t-t_B\simeq(2/f''(t_B))^{1/2}(f-f(t_B))^{1/2}. \tag{2.12}$$

Thus the singular part in (2.11) is written as

$$dt/df=\tfrac{1}{2}(2/f''(t_B))^{1/2}(f-f(t_B))^{-1/2}. \tag{2.13}$$

Putting this equation into (2.11) we have

$$J=2\,\mathrm{Re}\frac{1}{(\pi N)^{1/2}}\left(-\frac{1}{2f''(t_B^+)f(t_B^+)}\right)^{1/2}\sum_{m=0}^{\infty}\left(\frac{1}{f^2(t_B^+)N}\right)^m\frac{\Gamma(m+\frac{1}{2})\Gamma(2m+\frac{1}{2})}{\Gamma(2m+1)}. \tag{2.14}$$

The quantity $f^2(t_B)$ is

$$f^2(t_B)=F(t_B)-F(t_A)=-\frac{(1+16g)^{1/2}}{16g}-\frac{1}{2}\ln\frac{(1+16g)^{1/2}+1}{(1+16g)^{1/2}-1}\pm\frac{\pi}{2}i \qquad (n=1,2,\dots) \tag{2.15}$$

We thus obtain the large-order behaviour of (2.8) as

$$A_K\sim\frac{\sin\theta K}{\pi}\left(\frac{(1+16g)^{1/2}-1}{(16g)^{1/2}}\right)\frac{\Gamma(K+\frac{1}{2})}{\rho^K}K^{-1/2}, \tag{2.16}$$

with

$$\rho=\left(Z^2+\frac{\pi^2}{4}\right)^{1/2}, \qquad \theta=\cos^{-1}(Z/\rho),$$

$$Z=-\frac{(1+16g)^{1/2}}{16g}-\frac{1}{2}\ln\frac{(1+16g)^{1/2}+1}{(1+16g)^{1/2}-1}. \tag{2.17}$$

From (2.17) the quantity ρ takes the value $\pi/2$ in the limit $g\to\infty$. This result is checked by the $1/\sqrt{g}$ expansion

$$I=\frac{1}{4}g^{-N/4}\sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{-1}{2\sqrt{g}}\right)^{K}N^{N/4+K/2}\Gamma\left(\frac{N}{4}+\frac{K}{2}\right)\frac{1}{2^{N/2-1}\Gamma(N/2)}$$
$$\simeq\frac{\Gamma(N/4)N^{N/4}}{2^{N/2+1}\Gamma(N/2)}g^{-N/4}. \tag{2.18}$$

The large-order behaviour of the Γ function is given by

$$\ln\Gamma(Z)\simeq\sum\frac{(K-1)!}{(2\pi Z)^{K}}\frac{1}{\pi}\sin\frac{\pi}{2}K, \tag{2.19}$$

and this checks that (2.18) coincides with (2.16) in the infinite limit.

The large-order behaviour of the zero-dimensional $g\phi^4$ model is also calculated by the differential equation of the parabolic cylinder function. We have (Balian and Toulouse 1974)

$$I=\left(\frac{1}{2gN}\right)^{N/4}2^{-N/2}\,e^{N/32g}U\left(\frac{N-1}{2},\left(\frac{N}{8g}\right)^{1/1}\right), \tag{2.20}$$

where the parabolic cylinder functions $U(a,x)$ satisfy the differential equation

$$d^2U/dx^2=(\tfrac{1}{4}x^2+a)U. \tag{2.21}$$

For the large-a and large-x^2 case we obtain

$$U(a,x)=(a+\tfrac{1}{4}x^2)^{-1/4}\exp\left(\int_{x_0}^{x}(a+\tfrac{1}{4}t^2)^{1/2}\,dt\right). \tag{2.22}$$

The integral of the exponents leads to

$$\int_{\pm 2i\sqrt{a}}^{(N/8g)^{1/2}}\left(a+\frac{1}{4}t^2\right)^{1/2}dt=\frac{(1+16g)^{1/2}}{32g}+\frac{1}{2}\ln\frac{(1+16g)^{1/2}+1}{4\sqrt{g}}\pm\frac{\pi}{4}i. \tag{2.23}$$

Multiplying expression (2.23) by two we find the large-order behaviour (Dingle 1973), which coincides with expression (2.17).

As discussed by Bray (1974), it is convenient to consider the entropy instead of the partition function itself in order to obtain the numerical values of the perturbation. Denoting t as

$$t=1/2\sqrt{g} \tag{2.24}$$

the entropy S is written as

$$S=-(2/N)\,d(\ln Z)/dt. \tag{2.25}$$

It is easily shown that the quantity S satisfies the equation

$$S^2+tS=1+(2/N)\,dS/dt. \tag{2.26}$$

In the limit $N\to\infty$ we have

$$S_0=[(t^2+4)^{1/2}-t]/2. \tag{2.27}$$

S is expanded as

$$S=\sum_{m=0}^{\infty} C_m\left(\frac{2}{N}\right)^m, \qquad C_0=S_0. \tag{2.28}$$

This expansion is determined by the recurrence equation

$$\frac{\mathrm{d}C_{m-1}}{\mathrm{d}t}=(2C_0+t)C_m+\sum_{l=1}^{m-1} C_l C_{m-l}. \tag{2.29}$$

The term C_m is given by

$$C_m=\sum_{j=0}^{m} C(m,j)\frac{t^j}{(t^2+4)^{(2m+j-1)/2}}. \tag{2.30}$$

Solving the recurrence equation we obtained the first 60 terms for C_m on a CDC6499 computer. The values of ρ and θ in (2.17) are numerically verified.

2.3. Small-g behaviour (see appendix 1)

The small-g expansion in our zero-dimensional model is calculated as

$$\begin{aligned} IN^{N/2} &= \int_0^{\infty} \exp\left(-\frac{1}{2}x^2-\frac{g}{N}x^4\right)x^{N-1}\,\mathrm{d}x \\ &= \frac{2^{N-2}}{\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma(l+N/4)}{\Gamma(l+1)}\Gamma\left(l+\frac{1}{2}+\frac{N}{4}\right)\left(-\frac{16g}{N}\right)^l. \end{aligned} \tag{2.31}$$

The value of $-16g/N$ is consistent with our previous result. For the small-g case we can expand ρ in (2.17) as

$$\rho \simeq (1+16g)^{1/2}/16g-\tfrac{1}{2}\ln g+\mathrm{O}(1) \tag{2.32}$$

$$\theta \simeq 0+\mathrm{O}(g). \tag{2.33}$$

This behaviour of ρ is consistent with the result of the small-g expansion for fixed N.

3. N-vector model

In this section we investigate the one-dimensional problem, for which the exact solution is known. The model describes the N-component spin system, with fixed spin length. Therefore the model is very similar to the nonlinear σ model except that the spins are placed on a discrete linear chain. The Hamiltonian is given by

$$\mathscr{H}=-J\sum_i \boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1} \tag{3.1}$$

and

$$|\boldsymbol{S}_i|^2=\sum_{m=1}^{N} S_i^2(m)=N. \tag{3.2}$$

The partition function for the one-dimensional N-vector model containing an arbitrary number M of spins is easily derived (Stanley 1969) as

$$Z = \left[\left(\frac{N}{2}\frac{J}{kT}\right)^{1-N/2}\frac{N}{(2)}\mathrm{I}_{N/2-1}\left(\frac{NJ}{kT}\right)\right]^{M-1} \tag{3.3}$$

where I is a modified Bessel function. The large-order behaviour in the $1/N$ expansion comes mainly from the modified Bessel function. In equation (3.3) there is a Γ function. We know the large-order terms from this Γ function from equation (2.19). However, these terms are cancelled, as we will discuss below, by the behaviour of the modified Bessel function equation (3.4).

We consider the modified Bessel function $\mathrm{I}_\nu(\nu Z)$. The case with $\nu = N/2$ and $Z = 2J/kT$ is slightly different from equation (3.3). However, the difference is of the order $1/N$, and the large-order behaviour is not affected by this replacement. The modified Bessel function has the integral representation

$$\mathrm{I}_\nu(\nu Z) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\left(\frac{\nu Z}{2}\right)^\nu 4^\nu \, \mathrm{e}^{-\nu Z} S(\nu . Z), \tag{3.4}$$

with

$$S(\nu . Z) = \int_0^1 (u-u^2)^{\nu-1/2} \, \mathrm{e}^{2u\nu Z} \, \mathrm{d}u = \int_0^\infty \mathrm{e}^{-F(t)} \, \mathrm{d}t,$$

where

$$F(t) = (\nu+\tfrac{1}{2})t + (\nu-\tfrac{1}{2})\ln(1-\mathrm{e}^{-t}) + 2\nu Z \, \mathrm{e}^{-t}. \tag{3.5}$$

In the large-ν limit we have the saddle point equation

$$\partial F(t)/\partial t = 0. \tag{3.6}$$

We have two saddle points t_A and t_B which are given by the equations

$$\mathrm{e}^{-t_\mathrm{A}} = [Z-1+(1+Z^2)^{1/2}]/2Z, \qquad \mathrm{e}^{-t_\mathrm{B}} = [Z-1-(1+Z^2)^{1/2}]/2Z. \tag{3.7}$$

t_A has a real solution, and t_B is a complex number. By the same argument as in § 2.2 we have

$$F(t_\mathrm{B}) - F(t_\mathrm{A}) = 2\nu\left(\ln\frac{(1+Z^2)^{1/2}-1}{Z} + (1+Z^2)^{1/2} \pm \frac{\pi}{2}\mathrm{i}\right) = 2\nu\rho \, \mathrm{e}^{\pm \mathrm{i}\theta}, \tag{3.8}$$

where the amplitude ρ is given by

$$\rho = \left[\left(\ln\frac{(1+Z^2)^{1/2}-1}{Z} + (1+Z^2)^{1/2}\right)^2 + \frac{\pi^2}{4}\right]^{1/2} \tag{3.9}$$

and the phase θ is

$$\theta = \cos^{-1}\left(\ln\frac{(1+Z^2)^{1/2}-1}{Z} + (1+Z^2)^{1/2}\right)\rho^{-1}. \tag{3.10}$$

Therefore we have large-order behaviour at Kth order:

$$Z_K/N^K \propto (K-1)! \sin\theta K/\rho^K N^K. \tag{3.11}$$

From equation (3.4) we can easily derive the $1/N$ expansion as

$$\mathrm{I}_\nu(\nu Z) = \frac{1}{(2\pi\nu)^{1/2}} \frac{\mathrm{e}^{\nu\eta}}{(1+Z^2)^{1/4}}\left(1 + \sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right), \tag{3.12}$$

with

$$t=\frac{1}{(1+Z^2)^{1/2}}, \qquad \eta=(1+Z^2)^{1/2}+\ln\frac{(1+Z^2)^{1/2}-1}{Z}.$$

The terms $u_k(t)$ are determined by (Abramovitz)

$$u_{k+1}(t)=\tfrac{1}{2}t^2(1-t^2)u_k'(t)+\tfrac{1}{8}\int_0^t(1-5t^2)u_k(t)\,\mathrm{d}t, \qquad k=0,1,2,\ldots. \tag{3.13}$$

Writing $u_k(t)$ as

$$u_k(t)=\sum_{j=1}^{k+1}C(k,j)t^{k+2j-2}$$

we have the following recursion equation for $C(k,j)$,

$$C(k,j)=\left(\frac{k+2j-3}{2}+\frac{1}{8(k+2j-2)}\right)C(k-1,j)\delta_{\mathrm{A}}-\frac{k+2j-5}{2}C(k-1,j-1)\delta_{\mathrm{B}}$$
$$-\frac{5}{8(k+2j-2)}C(k-1,j-1)\delta_{\mathrm{B}}, \qquad (j=1,\ldots,k+1), \tag{3.14}$$

with

$$\delta_{\mathrm{A}}=\begin{cases}1 & 1\leqslant j\leqslant k\\ 0 & j=k+1,\end{cases} \qquad \delta_{\mathrm{B}}=\begin{cases}1 & 2\leqslant j\leqslant k+1\\ 0 & j=1,\end{cases} \qquad C(0,1)=1.$$

We calculated the first 60 terms for u_k for the various values of $Z=2J/kT$. Our theoretical value of equation (3.11) agrees well. We also observe for small Z ($\leqslant 0{\cdot}02$) the large-order behaviour of $u_k(t)$ which shows the inverse behaviour of $\Gamma(N/2)$ at large-order, and formula (3.11) holds for all positive values of Z.

It is interesting to consider the analytic continuation for $Z\rightarrow \mathrm{i}Z$ (the value of t can be continued to a value greater than one). This procedure transforms the modified Bessel function into the ordinary Bessel function $\mathrm{J}_\nu(\nu Z)$. However, the large-order behaviour of the $1/N$ expansion for $\mathrm{J}_\nu(\nu Z)$ is given by exactly the same recursion relation as equation (3.13) and (3.14), and it is easily found that the large-order behaviour becomes

$$Z_K\propto(K-1)!/\tilde{\rho}^K, \tag{3.15}$$

with

$$\tilde{\rho}=\ln\frac{-(1-Z^2)^{1/2}+1}{Z}+\sqrt{1-Z^2}.$$

For $Z=2J/kT$ large, the ρ in (3.7) can be expanded as

$$\rho\propto Z+(\pi^2/4-1)/(2Z)+\mathrm{O}(1/Z^2), \qquad \theta\propto\pi^2/2Z+\mathrm{O}(1/Z^2). \tag{3.16}$$

In the limit of infinite Z, the signs of the terms become the same. Therefore the series of the $1/N$ expansion seems to be non-Borel summable in the large-Z limit. This is related to the low-temperature expansion for the N-vector model. The modified Bessel function in (3.3) leads to

$$\mathrm{I}_{N/2-1}\left(\frac{NJ}{kT}\right)\sim\left(\frac{kT}{2\pi NJ}\right)^{1/2}\mathrm{e}^{NJ/kT}\sum A_K\left(\frac{kt}{NJ}\right)^K, \qquad A_K\sim K!2^{-K}. \tag{3.17}$$

The large-order behaviour of this low-temperature expansion does not show sign alternation.

4. One-dimensional $g\phi^4$ model

4.1. $1/N$ expansion

The one-dimensional $g\phi^4$ model is known to be equivalent for general values of N to the x^4 anharmonic oscillator (Balian and Toulouse 1974, Ferrell and Scalapino 1974).

Therefore is it convenient to consider the following quantum mechanical perturbation theory to obtain the $1/N$ expansion series:

$$\left[-\frac{1}{2}\left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}\right)+\frac{1}{2}r^2+\frac{g}{N}r^4-E\right]\psi=0. \tag{4.1}$$

Making the change of variables

$$\psi=r^{-(N-1)/2}\phi, \qquad r=\sqrt{N}\rho, \qquad E=\epsilon N \tag{4.2}$$

we have

$$\left(-\frac{1}{2}\frac{d^2}{d\rho^2}+\frac{N^2(1-1/N)(1-3/N)}{8\rho^2}+\frac{1}{2}N^2\rho^2+gN^2\rho^4-\epsilon N^2\right)\phi=0. \tag{4.3}$$

In the large-N limit the energy is given by

$$\epsilon_0=1/8\rho_c^2+\rho_c^2/2+g\rho_c^4. \tag{4.4}$$

The saddle point ρ_c is determined by

$$V(\rho)=1/8\rho^2+\rho^2/2+g\rho^4-\epsilon_0 \tag{4.5}$$

$$dV'(\rho_c)/d\rho=0. \tag{4.6}$$

The $1/N$ expansion for the ground-state energy is obtained by expanding the wavefunction around this saddle point. It is very convenient to transform the Schrödinger equation into a Riccati equation for numerical calculations:

$$u=\phi'/\phi, \qquad \rho=\rho_c+x/\sqrt{N} \tag{4.7}$$

$$[-\tfrac{1}{2}u'-\tfrac{1}{2}u^2+NV(\rho)+(-\tfrac{1}{2}+\tfrac{3}{8}N^{-1})\rho^{-2}]\phi=N(\epsilon-\epsilon_c)\phi. \tag{4.8}$$

The wavefunction u and the ground-state energy ϵ are written as

$$u=\sum_{k=1}^{\infty}\sum_{\substack{m=1\\ k-2m+2>0}}C(k,m)x^{k-2m+2}\frac{1}{N^{(k-1)/2}} \tag{4.9}$$

$$E=\epsilon_0 N+\sum_{k=1}^{\infty}\frac{\epsilon k}{N^{k-1}} \tag{4.10}$$

$$\epsilon_1=-\tfrac{1}{2}C(1,1)-\tfrac{1}{2}\rho_s^{-2}, \qquad \epsilon_2=-\tfrac{1}{2}C(3,2)+\tfrac{3}{8}\rho_s^{-2}, \qquad \epsilon_k=-\tfrac{1}{2}C(2k-1,k). \tag{4.11}$$

For the case where the coupling constant vanishes, we have the exact wavefunction. The wavefunction ψ and the ground -state energy E are written as

$$\psi(r)=e^{-r^2/2}, \qquad E=N/2, \tag{4.12}$$

and $\phi(y)$ becomes

$$\phi = N^{(N-1)/4} \exp\left(\frac{N-1}{2} \ln \rho - \frac{N}{2}\rho^2\right). \tag{4.13}$$

The saddle point ρ_s is $1/\sqrt{2}$. Expanding the exponent of ϕ around ρ_s we obtain

$$\phi = \left(\frac{N}{2}\right)^{(N-1)/4} e^{-N/4} e^{-x^2}\left(1 - \frac{x}{(2N)^{1/2}} + \frac{\sqrt{2}}{3\sqrt{N}}x^3 + O\left(\frac{1}{N}\right)\right). \tag{4.14}$$

Performing the numerical calculation for the ground-state energy we can obtain the asymptotic $1/N$ series. Making the truncation at the value where the absolute values of the terms become minimum we estimate the ground-state energy. For $N = 1$ the value of the ground-state energy has been investigated in detail by Hioe and Montroll (1975) and Seznec and Zinn-Justin (1978, private communication). For $N = 3$ the value of the first-excited-state energy of the $N = 1$ problem, since the singular part of the potential in (4.3) vanishes for $N = 3$ (tables 1 and 2).

Table 1. One-dimensional $g\phi^4$ model; the value of the ground-state energy is estimated by taking the partial sum of the $1/N$ expansion $\sum_{n=0}^{K_c} \epsilon_n (1/N)^{n-1}$.

g	K_c	$N = 1$	$(N = 1)$†	$N = 3$	$(N = 3)$‡
0·01	(13)	0·507 256 20	(0·507 256 20)	1·512 279 4	
0·05	(11)	0·532 649 39	(0·532 642 75)	1·557 667 3	
0·1	(8)	0·558 82	(0·559 14)	1·608 082 5	
0·5	(6)	0·697 469	(0·696 17)	1·895 52	(1·895 507)
1·0	(5)	0·805 583	(0·803 77)	2·137 34	(2·137 387)
1000	(4)	6·832 00	(6·6942)	16·667	

† Hioe and Montroll (1975).

‡ Seznec and Zinn-Justin (1978, private communication; from the first excitation value of $g = \frac{1}{6}, \frac{1}{3}$).

Table 2. The first column is the value of the partial sum of $g = 0·1$ in the $1/N$ expansion. The second column is the value of the small-g ($g = 0·1$) expansion (Bender and Wu 1969).

K	$\sum_{n=0}^{K} \epsilon_n \left(\frac{1}{N}\right)^{n-1}$	K	$\sum_{n=0}^{K} \hat{\epsilon}_n (0·1)^n$
1	0·564 345 6	1	0·575
2	0·557 444 8	2	0·548 75
3	0·559 966 8	3	0·569 562 5
4	0·558 637 6	4	0·545 433
5	0·559 525 3	5	0·581 243
6	0·558 820 6	6	0·517 260
7	0·559 458 0	7	0·650 234
8	0·558 823 4	8	0·335 752
9	0·559 493 8	9	1·169 29
10	0·558 781 2	10	−1·278 60
11	0·559 459 8	11	6·614 73
12	0·559 136 5	12	−21·124 0
13	0·558 063 2	13	84·440 6
14	0·563 663 9	14	−348·24
15	0·544 377 4	15	1 552·58

4.2. Instanton

When the coupling constant g is small and negative, there appears a real turning point x_t in the potential of equation (4.5). It leads to the instanton solution which connects the saddle point to the turning point, and we have $K!$ large-order behaviour for the $1/N$ expansion.

To consider the large-order behaviour we make the following change of variables:

$$N'^2 = (N-1)(N-3), \qquad g/N = g'/N'. \tag{4.15}$$

The potential in (4.3), which is expanded around x_c, turns out to be

$$V(x) = \tfrac{1}{8}x^{-2} + \tfrac{1}{2}x^2 + g'x^4 - \epsilon_0. \tag{4.16}$$

The large-order behaviour is calculated by the method which has been developed for the boson polynomial Lagrangian in one dimension (Brézin *et al* 1977):

$$E_K \simeq (K-1)!\rho^{-K}$$

$$\rho = 2\int_{x_c}^{x_t} (2\tilde{V}(x))^{1/2}\,dx = -\frac{1}{6g}\left(\frac{1}{x_c^4} + 8g'x_c^2\right)^{1/2} + \ln\left(\frac{(-1/8g')^{1/2} - (-1/8g')^{1/2} - (-1/8g' - x_c^6)^{1/2}}{x^3}\right), \tag{4.17}$$

in which we have normalised

$$V \text{ to } \tilde{V}(x) = V(2)/V''(x_c).$$

The turning point x_t is given by

$$x_t^2 = -2x_c^2 - 1/2g' = -1/8g'x_c^4. \tag{4.18}$$

For the small-g limit we have

$$\rho \simeq -(1/3g')(1+3g')^{1/2} + \tfrac{1}{2}\ln(-g'/4). \tag{4.19}$$

The value of $-1/3g'$ is consistent with the result of the small-g expansion (Appendix 2).

For g' positive we consider the large-order behaviour which is obtained by the analytic continuation of (4.17). In this case the turning point x_t becomes complex. A simple example of complex instanton has been investigated by Brézin *et al* (1977). The ρ is written as

$$\rho = A\,e^{i\theta},$$

where

$$A = [Z^2 + (\pi/2)^2]^{1/2}, \qquad \theta = \cos^{-1}(Z/A), \tag{4.20}$$

$$Z = -\frac{1}{6g}\left(\frac{1}{x_c^4} + 8gx_c^2\right)^{1/2} + \ln\frac{(1/8g + x_c^6)^{1/2} - (1/8g)^{1/2}}{x_c^3}.$$

For the large-g' limit the ground-state energy behaves due to the trivial scaling as

$$E \sim Cg^{1/3}. \tag{4.21}$$

The value of C is known to be 0·667 986 for $N=1$ and 1·659 66 for $N=3$. In our $1/N$ expansion we have

$$E = (0{\cdot}472\,47)Ng^{1/3} + 0{\cdot}283\,16\,g^{1/3} + \ldots. \tag{4.22}$$

This expansion is asymptotic, and the value of ρ is $\ln(\sqrt{2}+\sqrt{3})\pm\frac{1}{2}\pi i$. In the problem of the planar diagram, another $1/N$ expansion has been considered for the large-g limit by Brézin *et al* (1978).

The amplitude of the large-order behaviour may be derived as

$$E_K = \mathrm{Re}\left[-\frac{1}{2\pi^{3/2}}\frac{\Gamma(K+\frac{1}{2})}{\rho^{K+1/2}}y_+ \exp\int_0^{y_+} dy\left(\frac{1}{(2\tilde{V}_R(y))^{1/2}}-\frac{1}{y}\right)\right], \tag{4.23}$$

where the potential $\tilde{V}_R$ is normalised as $\frac{1}{2}y^2$ for small-y limit. We have from (4.23)

$$E_K = -\mathrm{Re}\left[\frac{\Gamma(K+\frac{1}{2})\exp[i(K+\frac{1}{2})\theta]}{\pi^{3/2}A^{K+1/2}}\left(\frac{x_t^2-x_c^2}{x_c}\right)\right] \tag{4.24}$$

The effect of the difference of N' and N in (4.15) for the large-order behaviour is considered as follows: the large-order behaviour is described as

$$\begin{aligned} E &\simeq N'\sum\frac{\Gamma(K+b)}{N'^K}(a(g'))^K c(g') \\ &\simeq N'\sum\frac{\Gamma(K+b)}{N^K}(a(g'))^K c(g)\exp\frac{2}{a(g)}\left(1-\frac{ga'(g)}{a(g)}\right). \end{aligned} \tag{4.25}$$

Therefore the quantities A and θ in (4.24) are not affected. The first 50 terms for the ground-state energy for arbitrary g was calculated numerically on a CDC6400 computer, and the values of A and θ were verified.

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Appendix 1

The small-g expansion in the zero-dimensional $g\phi^4$ model is given as

$$\begin{aligned} \ln I &= \ln\left[1+\sum_{k=1}^{\infty}\frac{\Gamma(k+N/4)}{\Gamma(k+1)}\frac{\Gamma(k+\frac{1}{2}+N/4)}{\Gamma(N/4)\Gamma(\frac{1}{2}+N/4)}\left(-\frac{16g}{N}\right)^k\right] \\ &= N(N+2)(g/N)+4N(N+2)(N+3)(g/N)^2 \\ &\quad -\tfrac{2}{3}N(N+2)(40N^2+287N+480)(g/N)^3+\dots. \end{aligned} \tag{A1.1}$$

The large-order behaviour of this expansion becomes

$$(\ln I)_k \simeq \frac{2^{N/2-1}}{\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(N/2)}\left(-\frac{16g}{N}\right)^k k^{N/2-1}\left(1+\frac{N(N-2)}{16k}+\dots\right). \tag{A1.2}$$

Appendix 2

We represent here the result of the small-g expansion for the one-dimensional $g\phi^4$ model with O(N) symmetry. The ground-state energy is calculated as

$$\left(-\frac{1}{2}\frac{d^2}{dx^2}-\frac{N-1}{2x}\frac{d}{dx}+\frac{1}{2}x^2+\frac{g}{N}x^4\right)\psi=E\psi. \tag{A2.1}$$

Denoting $u=\psi'/\psi$ we obtain

$$-\tfrac{1}{2}u'-\tfrac{1}{2}u^2-[(N-1)/2x]u+\tfrac{1}{2}x^2+(g/N)x^4=E. \tag{A2.2}$$

The function $u(x)$ is written as

$$u(x)=-\sum_{k=0}^{\infty}\sum_{M=1}^{k+1}C(k+1,M)\left(\frac{g}{N}\right)^k x^{2k-2M+1} \tag{A2.3}$$

and E is given by

$$\begin{aligned}E&=\sum_{k=0}^{\infty}\frac{N}{2}C(k+1,k+1)\left(\frac{g}{N}\right)^k\\&=\frac{N}{2}+\frac{N(N+2)}{4}\left(\frac{g}{N}\right)-\frac{N}{8}(N+2)(2N+5)\left(\frac{g}{N}\right)^2\\&\quad+\frac{N(N+2)(8N^2+43N+60)}{16}\left(\frac{g}{N}\right)^3+\ldots.\end{aligned} \tag{A2.4}$$

The asymptotic large-order behaviour of the ground-state energy E is given by (Banks *et al* 1973)

$$E=\sum_{k=0}^{\infty}a_k\left(\frac{g}{N}\right)^k \tag{A2.5}$$

$$A_k\simeq-\frac{1}{\pi}\frac{6^{N/2}}{\Gamma(N/2)}\Gamma\left(k+\frac{N}{2}\right)(-3)^k. \tag{A2.6}$$

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